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Vector calculus

Theorem → Prove that

$$\text{div}(\vec{u}+\vec{v}) = \text{div}\,\vec{u} + \text{div}\,\vec{v}$$

or $$\nabla\cdot(\vec{u}+\vec{v}) = \nabla\cdot\vec{u} + \nabla\cdot\vec{v}$$

Proof →

$$LHS = \text{Div}(\vec{u}+\vec{v}) = \nabla\cdot(\vec{u}+\vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\vec{u}+\vec{v})$$

$$= \vec{i}\cdot\frac{\partial}{\partial x}(\vec{u}+\vec{v}) + \vec{j}\cdot\frac{\partial}{\partial y}(\vec{u}+\vec{v}) + \vec{k}\cdot\frac{\partial}{\partial z}(\vec{u}+\vec{v})$$

$$= \vec{i}\cdot\left[\frac{\partial\vec{u}}{\partial x} + \frac{\partial\vec{v}}{\partial x}\right] + \vec{j}\cdot\left[\frac{\partial\vec{u}}{\partial y} + \frac{\partial\vec{v}}{\partial y}\right] + \vec{k}\cdot\left[\frac{\partial\vec{u}}{\partial z} + \frac{\partial\vec{v}}{\partial z}\right]$$

$$= \left(\vec{i}\cdot\frac{\partial\vec{u}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{u}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{u}}{\partial z}\right) + \left(\vec{i}\cdot\frac{\partial\vec{v}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{v}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{v}}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{u} + \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{v}$$

$$= \nabla\cdot\vec{u} + \nabla\cdot\vec{v}$$

$$= \text{Div}\,\vec{u} + \text{Div}\,\vec{v} = RHS$$ Proved

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Theorem

Prove that $\text{curl}(\vec{u}+\vec{v}) = \text{curl}\,\vec{u} + \text{curl}\,\vec{v}$

or

$$\nabla \times (\vec{u}+\vec{v}) = \nabla \times \vec{u} + \nabla \times \vec{v}$$

Proof

$$LHS = \text{curl}(\vec{u}+\vec{v}) = \nabla \times (\vec{u}+\vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (\vec{u}+\vec{v})$$

$$= \vec{i} \times \frac{\partial}{\partial x}(\vec{u}+\vec{v}) + \vec{j} \times \frac{\partial}{\partial y}(\vec{u}+\vec{v}) + \vec{k} \times \frac{\partial}{\partial z}(\vec{u}+\vec{v})$$

$$= \vec{i} \times \left[\frac{\partial \vec{u}}{\partial x} + \frac{\partial \vec{v}}{\partial x}\right] + \vec{j} \times \left(\frac{\partial \vec{u}}{\partial y} + \frac{\partial \vec{v}}{\partial y}\right)$$

$$+ \vec{k} \times \left(\frac{\partial \vec{u}}{\partial z} + \frac{\partial \vec{v}}{\partial z}\right)$$

$$= \left(\vec{i} \times \frac{\partial \vec{u}}{\partial x} + \vec{i} \times \frac{\partial \vec{v}}{\partial x}\right) + \left(\vec{j} \times \frac{\partial \vec{u}}{\partial y} + \vec{j} \times \frac{\partial \vec{v}}{\partial y}\right)$$

$$+ \left(\vec{k} \times \frac{\partial \vec{u}}{\partial z} + \vec{k} \times \frac{\partial \vec{v}}{\partial z}\right)$$

$$= \left(\vec{i} \times \frac{\partial \vec{u}}{\partial x} + \vec{j} \times \frac{\partial \vec{u}}{\partial y} + \vec{k} \times \frac{\partial \vec{u}}{\partial z}\right)$$

$$+ \left(\vec{i} \times \frac{\partial \vec{v}}{\partial x} + \vec{j} \times \frac{\partial \vec{v}}{\partial y} + \vec{k} \times \frac{\partial \vec{v}}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{u}$$

$$+ \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{v}$$

$$= \nabla \times \vec{u} + \nabla \times \vec{v}$$

$$= \text{curl}\,\vec{u} + \text{curl}\,\vec{v} = RHS$$

Proved